International Journal of Theoretical Physics, Vol. 15, No. 5 (1976), pp. 339–348

Nonperturbative Green's Function Approach to Quantum Field Theory and Arbitrariness of Solutions

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Received: 10 *June* 1975

Abstract

Green's function equations are considered for interacting spinor and (pseudo)scalar fields with interactions $g\bar{\psi}\gamma\psi\phi + \frac{1}{4}\lambda\phi^4$. These equations do not determine higher many-point functions if two-point functions are given as "input." If vertex parts are given as input, two-point functions are determined but higher many-point functions are not determined.

1. *Introduction*

In this paper we consider the descending and ascending problems in the Green's function approach to quantum field theory formulated in our previous paper (Yoshimura, 1975) for interacting spinor and (pseudo)scalar fields with interactions

$$\mathscr{L}_{\text{int}} = -g\bar{\psi}\gamma\psi\phi - \lambda\phi^4 \quad (\gamma = I \text{ or } \gamma_5)$$

The main questions are whether one can determine the vertex parts Γ and Λ without resorting to perturbation theory if two-point functions were known or substituted by model functions (ascending problem), whether one two-point function and one vertex part can be determined if another two-point function and another vertex part are given (mixed problem), and whether two-point functions can be determined if the vertex parts are given (descending problem). For these problems the relevant equations are the following Schwinger–Dyson equations:

$$[i\gamma p - m - g\, [\![G^* \mathfrak{G}^* \Gamma]\!](p)]\, G(p) = 1 \tag{1.1a}$$

$$[k^2 - \mu^2 - g\, [\![\gamma G^* \Gamma^* G]\!]\,(k) - \lambda\, [\![\mathfrak{G}^{**3} {*} \Lambda]\!]\,(k)]\, \mathfrak{G}\,(k) = 1 \tag{1.1b}$$

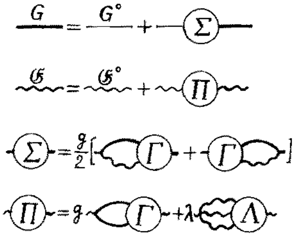


Figure 1.

where G and $\mathfrak{G}$ are two point functions (in Heisenberg representation) of the spinor and (pseudo)scalar fields, respectively, and Γ and Λ are the vertex parts (See Figure 1). For the meaning of $[\![\cdots]\!]$ see Appendix.

In Section 2, we consider the descending problem in the case $\lambda = 0$. In Section 3, we consider the descending problem with $\lambda = 0$. In Section 4, we consider the ascending problem with $\lambda \neq 0$. In Section 5, we consider the mixed problem with $\lambda \neq 0$. In Section 6, we consider the descending problem with $\lambda \neq 0$. Section 7 is devoted to discussion and remarks.

2. *Ascending Problem in Pure Yukawa Type Theory*

In this Section we consider the question whether one could determine the vertex part $\Gamma(p, q)$ if one of the two-point functions G or $\mathfrak{G}$ were given and if $\lambda = 0$.

In this case one can eliminate $\mathfrak{G}$ and write equation (1.1) as follows:

$$\sigma(p) - gG^0 [\![(G^0 + \sigma) * [(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0 + \sigma)^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!]\{(G^0)^{-1} + g[\![(G^0 + \sigma) * [(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0 + \sigma)^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!]\}^{-1} \equiv \Omega[\Gamma](p) = 0 \tag{2.1}$$

where G^0 and $\mathfrak{G}^0$ are the free propagators. (Symmetrizations are to be made as in Figure 1.) It should be noticed that nonlinearity of equation (2.1) is marginal in the sense that the nonlinearity comes from convolutions and the algebraic analog of equation (2.1) is linear.

Now we try to find solutions of this equation in a reflexive Banach space $\mathfrak{B}$ of functions of two four-momenta. We define a norm in the Banach space $\mathfrak{B}$ as follows:

$$\|\Gamma\|_{\mathfrak{B}} = c_1 \sup |\Gamma(p, q)| + c_2 [\textstyle\int d^4p d^4q \, |\Gamma(p, q)|^{\kappa}]^{1/\kappa}, \quad \kappa > 2 \tag{2.2}$$

The second term on the right-hand side makes the unit ball in $\mathfrak{B}$ uniformly convex so that reflexivity is guaranteed, which is necessary for the existence of the sequence $\{\Theta_n\}$ to be defined below (See Janko, 1968).

It is convenient to normalize Γ so as to make $\Gamma(p, p)|_{p^2 = m^2} = 1$ (or γ_5), i.e., our g corresponds to g^2 in more conventional normalization. For a zeroth approximation Γ_0 to have a finite norm, $\Gamma_0(p, q)$ must behave asymptotically as follows:

$$\Gamma_0(p,q) \leqslant c_3 \, [\max(p^2, q^2, (p-q)^2)]^{-\kappa/4} \tag{2.3}$$

It should be noticed that for the descending problem to have a solution, $\Gamma(p,q)$ need not behave like (2.3) but

$$\Gamma(p,q) \leqslant c_4 \, \frac{|p^2|^\alpha |q^2|^\alpha |(p-q^2)^2|^\alpha}{[|p^2| + |q^2| + |(p-q)^2|]^{2\alpha+\beta}}, \quad \alpha, \beta > 0 \tag{2.4}$$

(See, Yoshimura, 1975.)

We define the following norm in the Banach space $\mathfrak{C}$ of candidates for σ:

$$\| \sigma \|_{\mathfrak{C}} = c_5 \sup |\sigma(p)| + c_6 \sup |p\sigma(p)| + c_7 [\int d^4p \, |\sigma(p)|^\mu]^{1/\mu} \tag{2.5}$$

Now we take a σ with norm $\| \sigma \|_{\mathfrak{C}} = O(g)$ as input and try to find a Γ satisfying equation (2.1). For the ascending problem it is more convenient to write equation (2.1) in the following form:

$$\sigma(G^0)^{-1}G^{-1} - g[\![\Gamma * G\gamma * \mathfrak{G}^0]\!] - g^2 \left[\!\!\left[\Gamma * G\gamma * \left(\frac{\mathfrak{G}^0 [\![\Gamma * G^{**2}\gamma]\!]}{(\mathfrak{G}^0)^{-1} - g[\![\Gamma * G^{**2}\gamma]\!]}\right)\right]\!\!\right] \equiv \Xi[\Gamma] = 0 \tag{2.6}$$

where $G = G^0 + \sigma$. Let us write $\Gamma = \Sigma_n \Theta_n$ and try to find Θ_h as follows: The condition to be satisfied by Θ_0 is

$$g[\![\Theta_0 * G\gamma * \mathfrak{G}^0]\!] = \sigma(G^0)^{-1}G^{-1}, \quad \| \Theta_0 \|_{\mathfrak{B}} = O(g^0) \tag{2.7a}$$

Though we do not have a rigorous proof, we assume that at least one Θ_0 satisfying this condition exists in our reflexive (uniformly convex) Banach space $\mathfrak{B}$. Then the next step is to find Θ_1 that satisfies the condition:

$$[\![\Theta_1 * G\gamma * \mathfrak{G}^0]\!] = g[\![\Theta_0 * [\![\Theta_0 * G^{**2}\gamma]\!](\mathfrak{G}^0)^2 * G\gamma]\!], \quad \| \Theta_1 \|_{\mathfrak{B}} = O(g) \tag{2.7b}$$

Then the condition to be satisfied by Θ_2 is

$$\begin{aligned} [\![\Theta_2 * G\gamma * \mathfrak{G}^0]\!] &= g[\![\Theta_1 * G\gamma * (\mathfrak{G}^0)^2 [\![\Theta_0 * G^{**2}\gamma]\!]]\!] \\ &+ g[\![\Theta_0 * G\gamma * (\mathfrak{G}^0)^2 [\![\Theta_1 * G^{**2}\gamma]\!]]\!] \\ &+ g^2 \left[\!\!\left[\Theta_0 * G\gamma * \left(\frac{(\mathfrak{G}^0)^2 [\![\Theta_0 * G^{**2}\gamma]\!]^2}{(\mathfrak{G}^0)^{-1} - g[\![\Theta_0 * G^{**2}\gamma]\!]}\right)\right]\!\!\right] \end{aligned} \tag{2.7c}$$

and so on.

Then for sufficiently small g one can expect

$$\| \Gamma^* - \Gamma \| = O(g^{n+1}) \tag{2.8}$$

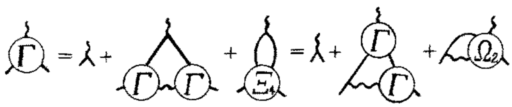
i.e., the procedure converges to a solution of equation (2.1).

For uniqueness of the solution one has to prove the uniqueness of the sequence $\{\Theta_n\}$ under reasonable auxiliary conditions such as symmetry under permutation of variables, etc. Unfortunately this problem is beyond the scope of the present article. (It is unlikely that Γ is unique.) Actual search of the sequence $\{\Theta_n\}$ is a very difficult problem. For the time being we must be content with existential arguments. Even if one had solved the descending problem with a given Γ with asymptotic behavior (2.4) and found σ, the scheme (2.7) does not converge to the original Γ whatever sequence $\{\Theta_n\}$ one takes, because the original Γ is not in the reflexive Banach space $\mathfrak{B}$.

What is the situation if $\mathfrak{G}$ is given? In this case one cannot eliminate G and consequently one cannot proceed with any presently available methods, though solutions may exist. The following question cannot be answered either: Can one prove or disprove the existence of Γ if both G and $\mathfrak{G}$ are arbitrarily given?

The ascending problems from a given set of $G, \mathfrak{G}, \Gamma$ are linear. The relevant equations are

$$\Gamma = \gamma + g[\![\Gamma^{**2} * G^{**2}\gamma * \mathfrak{G}]\!] + g[\![\gamma G^{**2} * \Xi_4]\!] \tag{2.9a}$$



$$= \gamma + g[\![\gamma G^{**2} * \mathfrak{G} * \Gamma]\!] + g[\![\gamma G * \mathfrak{G} * \Omega_2]\!] \tag{2.9b}$$

with unknown Ξ_4 and Ω_2, where Ξ_n and Ω_m are amputated many-point functions with n formion buds and with two fermion buds and m boson buds, respectively.

As the equation

$$[\![\gamma G^{**2} * \Xi_4']\!] = 0 \tag{2.10}$$

with unknown Ξ_4' has a continuum of solutions, if a solution of equation (2.9) exists, there are infinitely many solutions.

The main difficulty of the ascending problem is arbitrariness of solutions, rather than nonexistence of solutions. The question remaining is whether one can impose the unitarity and causality as auxiliary conditions so as to choose the solution, or whether those conditions are built into equation (1.1).

3. *Descending Problem with* $\lambda = 0$

In this Section, we consider the problem of how to determine σ if Γ is known or substituted by a model function. The equation to be considered is equation (2.1) with given Γ and unknown σ. We write this equation as follows:

$$(I - \Phi)[\sigma] \equiv \Psi[\sigma] = 0 \tag{3.1}$$

For this equation, the Frechet derivatives are

$$\Psi'[\sigma';\sigma] = I\sigma - \{g[\![\gamma\sigma * [(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!]$$
$$+ 2g^2[\![\gamma(G^0+\sigma') * \left(\frac{[\![\sigma * \gamma(G^0+\sigma') * \Gamma]\!]}{[(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!]]^2}\right)$$
$$\times * \Gamma]\!]\}\{(G^0)^{-1} - g[\![(G^0+\sigma') *$$
$$\times [(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!]\}^{-2} \tag{3.2}$$

$$\Psi''[\sigma';\sigma_1,\sigma_2] = \{2g^2[\![\sigma_1\gamma * [\![\sigma_2 * \gamma(G^0+\sigma') * \Gamma]\!](\mathfrak{G}^0)^2 * \Gamma]\!] + (\sigma_1 \leftrightarrow \sigma_2)$$
$$-2g^2[\![\gamma(G^0+\sigma') * [\![\sigma_1 * \gamma\sigma_2 * \Gamma]\!][(\mathfrak{G}^0)^{-1}$$
$$-g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!] - 2g^2[\![\gamma\sigma_2 * [\![(G^0+\sigma')$$
$$\times * \gamma\sigma_1 * \Gamma]\!][(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!]$$
$$-(\sigma_1 \leftrightarrow \sigma_2)\}\{(G^0)^{-1} - g[\![\gamma(G^0+\sigma') * [(\mathfrak{G}^0)^{-1}$$
$$-g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!]]^{-1} * \Gamma]\!]\}^{-2} + O(g^3) \tag{3.3}$$

so that we have the following estimates when $g \ll 1$:

$$\begin{aligned}
&\|(\Psi'[0;\cdot])^{-1}\| \leqslant B_0 = O(g^0)\\
&\|(\Psi'[0;\cdot])^{-1}\Psi[0]\| \leqslant \eta_0 = O(g)\\
&\|\Psi''[\sigma';\cdot,\cdot]\| \leqslant K = O(g^2)\ \forall \sigma' \in \mathfrak{S} = [\sigma \mid \|\sigma\| < (1-\sqrt{1-h_0})\, h_0^{-1}]\\
&h_0 \equiv B_0 K \eta_0 = O(g) < \tfrac{1}{2}
\end{aligned} \tag{3.4}$$

Therefore the Newton-Kantorovich scheme converges. If one fixes a Γ such that $[\![\gamma G * \mathfrak{G} * \Gamma]\!]$ and $[\![\gamma(G^0)^{**2} * \Gamma]\!]$ are finite for finite p^2, one finds that

$$\Psi'[0;\sigma] \in \mathfrak{C},\ \Psi''[0;\sigma,\sigma'] \in \mathfrak{C}\ \forall \sigma, \sigma' \in \mathfrak{S} \tag{3.5}$$

Some remarks are in order. If a theory has three particle thresholds in the self-energy parts, unrenormalised self-energy parts have threshold behavior $(p^2 - p_{\text{thr}}{}^2)^{1/2}$. Therefore the renormalization procedure

$$\int_{m_r^2}^{p^2} d(p'^2) \int_{m_r^2}^{p'^2} d(p''^2) \frac{d^2}{d(p''^2)^2}$$

must be carried out with use of pseudofunctions, so that the positive definiteness of Im G and Im $\mathfrak{G}$ is violated. As the signs of $\lim\limits_{p^2\to\infty} \text{Re}\ \gamma p G(p)$ and $\lim\limits_{k^2\to\infty} \text{Re}\ k^2 \mathfrak{G}(k^2)$ are not inverted, the Källén-Lehmann theorem holds but in a weaker sense.

4. *Ascending Problem with* $\lambda \neq 0$

For this problem, we begin with the following equations:

$$\sigma[(G^0)^{-1} - g[\![\gamma G * \mathfrak{G} * \Gamma]\!]] - gG^0[\![\gamma G * \mathfrak{G} * \Gamma]\!] = 0 \tag{4.1}$$

$$\rho[(\mathfrak{G}^0)^{-1} - g[\![\gamma G^{**2}\Gamma]\!] - \lambda[\![\mathfrak{G}^{**3} * \Lambda]\!]]$$
$$-g\mathfrak{G}^0[\![G^{**2} * \Gamma]\!] - \lambda\mathfrak{G}^0[\![\mathfrak{G}^{**3} * \Lambda]\!] = 0 \quad (4.2)$$

If G and $\mathfrak{G}$ are given arbitrarily, equation (4.1) is a linear functional equation for Γ. By functional equation, we mean an equation that involves an operator that maps an unknown function to a function of fewer variables or constants. This equation has a continuum of solutions, so let us pick one. Then equation (4.2) becomes a linear functional equation for Λ, so that one cannot determine Γ and Λ from these equations. The situation is quite similar to that of Section 2.

5. *Mixed Problems*

In this section we consider the problem with one self-energy part and one vertex part given.

If G and Γ are given, equation (4.1) becomes a linear operator equation for $\mathfrak{G}$ and determines $\mathfrak{G}$, and consequently equation (4.2) becomes a linear functional equation for Λ, which does not determine Λ uniquely.

If $\mathfrak{G}$ and Γ are given, equation (4.1) becomes a nonlinear operator equation for G, for which the arguments of Section 3 can be applied, i.e., G is determined. Then equation (4.2) becomes a linear functional equation for Λ, which does not determine Λ uniquely.

If σ and Λ are given, we cannot say anything because no known scheme is applicable. In other words, we cannot tell whether a given pair σ, Λ is compatible. The situation is similar when ρ and Λ are given.

6. *Descending Problem with* $\lambda \neq 0$

Now let us consider how the situation of Section 3 is changed if a $\lambda\phi^4$ term is present. In this case, one cannot eliminate ρ from equation (2.1), so that one has to deal with the direct product of the Banach space $\mathfrak{C}$ of candidates of σ and the Banach space $\mathfrak{D}$ of candidates of ρ. We define the norm in this direct product space $\mathfrak{C}\otimes\mathfrak{D}$ as follows:

$$\|(\sigma, \rho)\| = \|\sigma\|_{\mathfrak{C}} + \|\rho\|_{\mathfrak{D}} \quad (6.1a)$$

$$\|\sigma\|_{\mathfrak{C}} = \epsilon_1 \sup|p\sigma_1(p^2)| + \epsilon_2 \sup|\sigma_1(p^2)|$$
$$+ \epsilon_3 \sup|p^2\sigma_2(p^2)| + \epsilon_4 \sup|\sigma_2(p^2)|$$
$$+ \epsilon_5[\textstyle\int d^4p\,|\sigma_1(p^2)|^{\zeta_1}]^{1/\zeta_1} + \epsilon_6[\int d^4p\,|\sigma_2(p^2)|^{\zeta_2}]^{1/\zeta_2}$$
$$[\sigma(p) = \gamma p\sigma_1(p^2) + \sigma_2(p^2), \quad \zeta_1 > 4, \quad \zeta_2 > 2] \quad (6.1b)$$

$$\|\rho\|_{\mathfrak{D}} = \eta_1 \sup|p^2\rho(p^2)| + \eta_2 \sup|\rho(p^2)|$$
$$+ \eta_3[\textstyle\int d^4p\,|\rho(p^2)|^{\zeta_3}]^{1/\zeta_3} \qquad (\zeta_3 > 2) \quad (6.1c)$$

Then one can generalize the Newton-Kantorovich scheme as follows:

Rewrite equations (4.1) and (4.2) symbolically

$$[I - \Phi]\,[(\sigma, \rho)^T] = 0 \tag{6.2}$$

where Φ is a map from $\mathfrak{C} \otimes \mathfrak{D}$ into itself, whose explicit form is

$$\Phi[(\sigma,\rho)^T] = \left(\frac{gG^0 [\![\gamma(G^0+\sigma) * (\mathfrak{G}^0+\rho) * \Gamma]\!]}{(G^0)^{-1} - g[\![\gamma(G^0+\sigma) * (\mathfrak{G}^0+\rho) * \Gamma]\!]}, \right.$$
$$\left. \frac{\mathfrak{G}^0 (g[\![\gamma(G^0+\sigma)^{**2} * \Gamma]\!] + \lambda[\![(\mathfrak{G}^0+\rho)^{**3} * \Lambda]\!])}{(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma)^{**2} * \Gamma]\!] - \lambda[\![(\mathfrak{G}^0+\rho)^{**3}\Lambda]\!]} \right)^T \tag{6.3}$$

Define the sequence $\{(\sigma_m, \rho_m)\}$ by the following formula:

$$(\sigma_{n+1}, \rho_{n+1})^T = (\sigma_n, \rho_n)^T - \{I - H[(\sigma_n, \rho_n); \cdot\,]\}^{-1} [I - \Phi]\,[(\sigma_n, \rho_n)^T] \tag{6.4}$$

where H is a supermatrix

$$H[(\sigma', \rho')] = \begin{bmatrix} H_{11}[\sigma', \rho'] & H_{12}[\sigma', \rho'] \\ H_{21}[\sigma', \rho'] & H_{22}[\sigma', \rho'] \end{bmatrix} \tag{6.5}$$

$$H_{11}[\sigma', \rho'] = \frac{g}{2} \frac{[\![\gamma \cdot * (\mathfrak{G}^0+\rho') * \Gamma]\!] + [\![\Gamma * (\mathfrak{G}^0+\rho') * \cdot \gamma]\!]}{(G^0)^{-1} - g[\![\gamma(G^0+\sigma') * (\mathfrak{G}^0+\rho') * \Gamma]\!]} + O(g^2) \tag{6.6a}$$

$$H_{12}[\sigma', \rho'] = \frac{g}{2} \frac{[\![\gamma(G^0+\sigma') * \cdot * \Gamma]\!] + [\![\Gamma * \cdot * (G^0+\sigma')\gamma]\!]}{(G^0)^{-1} - g[\![\gamma(G^0+\sigma') * (\mathfrak{G}^0+\rho') * \Gamma]\!]} + O(g^2) \tag{6.6b}$$

$$H_{21}[\sigma', \rho'] = \frac{2g[\![(G^0+\sigma')\gamma * \cdot * \Gamma]\!]}{(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!] - \lambda[\![(\mathfrak{G}^0+\rho')^{**3} * \Lambda]\!]} + O(g^2, \lambda^2) \tag{6.6c}$$

$$H_{22}[\sigma', \rho'] = \frac{3\lambda[\![(\mathfrak{G}^0+\rho')^{**2} * \cdot * \Lambda]\!]}{(\mathfrak{G}^0)^{-1} - g[\![\gamma(G^0+\sigma')^{**2} * \Gamma]\!] - \lambda[\![(\mathfrak{G}^0+\rho')^{**3} * \Lambda]\!]} + O(g^2, \lambda^2) \tag{6.6d}$$

If asymptotically

$$|\Gamma(p,q)| \leqslant c_1' \frac{|p^2|^\alpha |q^2|^\alpha |(p-q)^2|^\beta}{(|p^2| + |q^2| + |(p-q)^2|)^{2\alpha+\beta}}, \quad \alpha, \beta > 0 \tag{6.7}$$

$$|\Lambda(p_1, p_2, p_3, p_1+p_2-p_3)| \leqslant c_2' \frac{|p_1^2|^\gamma |p_2^2|^\gamma |p_3^2|^\gamma |(p_1+p_2-p_3)^2|^\gamma}{(|p_1^2| + |p_2^2| + |p_3^2| + |(p_1+p_2-p_3)^2|)^{4\gamma}},$$
$$\gamma > 0 \tag{6.8}$$

and $g \ll 1, \lambda \ll 1$, then

$$\begin{aligned} &\| I + H(0,0) \| = O(1), \qquad &&\| [I + H(0,0)]^{-1} \| = O(1) \\ &\| \Phi''(0,0) \| = O(g^2, \lambda^2, g\lambda), \qquad &&\| \Phi(0,0) \| = O(g, \lambda) \end{aligned} \tag{6.9}$$

so that the Newton-Kantorovich scheme converges. Because the Lipschitz condition

$$\| H(\sigma_1, \rho_1) - H(\sigma_2, \rho_2) \| < K \| (\sigma_1, \rho_1) - (\sigma_2, \rho_2) \| \ \forall \ (\sigma_i \rho_i) \in S(0, 2r_0) \tag{6.10}$$

is satisfied with $K = O(g^2, \lambda^2)$ the solution (σ^*, ρ^*) is unique in the ball $S\{0, 2r_0\}$ where r_0 is the upper bound of $\| [I - H(0,0)]^{-1} (I - \Phi)(\sigma, \rho) \|$:

$$\| [I - H(0,0)]^{-1} (I - \Phi)(\sigma, \rho) \| < r_0 \ \forall \ (\sigma, \rho) \in S\{0, r_0\} \tag{6.11}$$

This, of course, does not mean that the solution of the descending problem is globally unique. On the other hand, it is an interesting feature that the vertex parts Γ and Λ can be given independently.

7. *Concluding Remarks*

As has been seen above, the main difficulty in the nonperturbative approach is the arbitrariness rather than the nonexistence of solutions. One cannot ascend to the original Γ, starting from the σ^* obtained as a solution to the descending problem with a Γ with the asymptotic behavior (2.4), because of the lack of reflexivity of the Banach space containing functions with asymptotic behavior (2.4). The procedure (2.7) "converges" to a function Γ' within the Banach space $\mathfrak{B}$, i.e., there are many Γ corresponding to a σ^*. This arbitrariness is not surprising. Fixing G and $\mathfrak{G}$ is not sufficient to determine a solution of the functional differential equations that generate the equations we have to deal with. This can be easily seen if one considers the discrete analog of our functional equations. Let us consider the following equations:

$$\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_{n-1}} - m \frac{\partial}{\partial x_n} - g \frac{\partial^2}{\partial x_n \partial y_n} \right) \mathscr{G}(x_1, \ldots, y_1, \ldots) = 0 \tag{7.1a}$$

$$\left(\frac{\partial}{\partial y_{n+1}} - 2 \frac{\partial}{\partial y_n} + \frac{\partial}{\partial y_{n-1}} - \mu \frac{\partial}{\partial y_n} - g \frac{\partial^2}{\partial x_n{}^2} - \lambda \frac{\partial^3}{\partial y_n{}^3} \right) \mathscr{G}(x_1, \ldots, y_1, \ldots) = 0 \tag{7.1b}$$

Then our G and $\mathfrak{G}$ correspond to

$$\frac{\partial^2}{\partial x_n \partial x_m} \mathscr{G}(x_1, \ldots, y_1, \ldots) \bigg|_{x_1 = \cdots = y_1 = \cdots = 0}$$

and

$$\frac{\partial^2}{\partial y_n \partial y_m} \mathscr{G}(x_1, \ldots, y_1, \ldots) \bigg|_{x_1 = \cdots = y_1 = \cdots = 0}$$

respectively. It is obvious that these quantities are not sufficient to determine the solution of equations (7.1). Boundary conditions must be given on a manifold of higher dimensions, not at a single point $x_1 = x_2 = \cdots = y_1 = y_2 = \cdots = 0$. The situation is not much different even if some of the higher derivatives at a single point are fixed. The latter situation corresponds to fixing vertex parts and/or some higher many-point functions. Moreover equations (7.1) are singular at $g = 0$, $\lambda = 0$ in the sense that the terms with the highest derivatives vanish when $g = 0$, $\lambda = 0$. Therefore it is meaningless to try to find a solution to equations (7.1) as power series in g and λ.

Another interesting question is whether the functions Θ_n must be symmetric under charge conjugation. We do not know the answer to this question, unfortunately.

The above arguments can be repeated for the so-called unrenormalizable interactions with minor changes.

Appendix: Renormalization Procedure

Though the self-energy parts do not diverge for finite values of momentum when Γ behaves as (2.3) or (2.4), the self-energy parts ought to be renormalized. As is mentioned in our previous paper (Yoshimura, 1975), the conventional renormalization by means of δm and Z is not suited to our equations, so we renormalize by the following substitutions:

$$\int d^4k\gamma G(p-k)\mathfrak{G}(k)\Gamma(p-k,p)$$
$$\to \int_{m^2}^{p^2} d(p'^2) \int_{m^2}^{p'^2} d(p''^2)\frac{d^2}{d(p''^2)^2}\int d^4k\gamma G(p''-k)\mathfrak{G}(k)\Gamma(p''-k,p'')$$
$$\equiv [\![\gamma G * \mathfrak{G} * \Gamma]\!](p) \tag{A1}$$

$$\int d^4p \operatorname{Tr} \gamma G\left(p-\frac{k}{2}\right)\Gamma\left(p-\frac{k}{2},p+\frac{k}{2}\right)G\left(p+\frac{k}{2}\right)$$
$$\to \int_{\mu^2}^{k^2} d(k'^2) \int_{\mu^2}^{k'^2} d(k''^2)\frac{d^2}{d(k''^2)^2}\int d^4k \operatorname{Tr} \gamma G\left(p-\frac{k''}{2}\right)\Gamma\left(p-\frac{k''}{2},p+\frac{k''}{2}\right)$$
$$\times G\left(p+\frac{k''}{2}\right) \equiv [\![\gamma G^{**2} * \Gamma]\!] \tag{A2}$$

(The * is a shorthand for convolution. For orders of convolutions, one has to refer to diagrams.)

The vertex parts are regularized by the substitutions

$$[\gamma G^{**2} * \Gamma^{**2} * \mathfrak{G}] \to D^{-1}D[\gamma G^{**2} * \Gamma^{**2} * \mathfrak{G}] \equiv$$
$$[\![\gamma G^{**2} * \Gamma^{**2} * \mathfrak{G}]\!](p_1,p_2) \tag{A3}$$

etc., where

$$D = p_{i\mu}\,\partial/\partial p_{i\mu} \tag{A4}$$

and the operation D^{-1} is to be carried out in such a way that the resultant expressions are equal to zero when $p_1 = p_2 = 0$ (Taylor, 1968).

References

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